J. Fluid Mech. (1978), *vol.* 84, *part* 1, *pp.* 181–191
Printed in Great Britain

An axisymmetric Boussinesq wave

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(Received 24 December 1976 and in revised form 27 June 1977)

An axisymmetric gravity wave, for which each of nonlinearity, dispersion and radial spreading is weak but significant, is determined as a similarity solution with slowly varying amplitude $\frac{3}{4}\mathscr{S}a$ and length scale l, where $a/d \propto (r/d)^{-\frac{2}{3}}$, $l/d \propto (r/d)^{\frac{1}{3}}$, r is the radius, d is the depth, and $\mathscr{S}$ is the family parameter of the solutions. It is shown that the free-surface displacement $\eta(r,t)$ is either a wave of elevation ($\eta \geqslant 0$) or a wave of depression ($\eta \leqslant 0$) and that $(|\eta|/a)^{\frac{1}{2}}$ satisfies a Painlevé equation that is a nonlinear generalization of Airy's equation. Representative numerical solutions and asymptotic approximations for small and large $\mathscr{S}$ are presented. It is shown that the similarity solution conserves energy but not mass, in consequence of which (in order to obtain a complete solution to a well-posed initial-value problem) it must either be accompanied by some other component or components or be driven by a source (or sink) in some interior domain in which the implicit restriction $r \gg d$ is violated. A linear model is developed that is valid for $r \lesssim d$ and compensates for the mass defect of, and matches, the nonlinear similarity solution for $|\mathscr{S}| \ll 1$.

1. Introduction

I consider here an axisymmetric similarity solution of the two-dimensional Boussinesq equations for gravity waves in water of uniform depth on the hypothesis that each of nonlinearity, dispersion and geometrical spreading is weak but of uniform relative strength – or, equivalently (see below), that nonlinearity and dispersion maintain a uniform balance and that the energy of the wave is conserved.

Let a and l be amplitude and length scales, d the quiescent depth and r the radius. Nonlinearity, dispersion and spreading are measured by a/d, $(d/l)^2$ and l/r, respectively, and radial invariance of their relative magnitudes implies $a/d \propto (r/d)^{-\frac{2}{3}}$ and $l/d \propto (r/d)^{\frac{1}{3}}$. A convenient (for the subsequent development) scaling is

$$a/d = \tfrac{4}{3}(2r/d)^{-\frac{2}{3}}, \quad (d/l)^2 = (2r/d)^{-\frac{2}{3}}, \tag{1.1 a, b}$$

and the corresponding similarity solution may be posited in the form†

$$\eta(r,t) = a(r)\,N(z), \quad z = (r-\tau)/l(r), \quad \tau = (gd)^{\frac{1}{2}}t, \tag{1.2 a, b, c}$$

where η is the free-surface displacement, t is the time, z is a phase variable, and N describes the wave profile. The (dimensionless) inverse wave speed is given by

$$(gd)^{\frac{1}{2}}/c \equiv (\partial\tau/\partial r)_z = l(\partial z/\partial r)_\tau = 1 - \tfrac{1}{3}(lz/r), \tag{1.3}$$

† The similarity formulation (1.2) remains formally valid under the transformation $\tau \to -\tau$ and may be used to describe incoming waves; cf. Cumberbatch (1977). Numerical solutions for a cylindrically converging 'solitary wave' are given by Chwang & Wu (1976), but they do not invoke similarity.

which approximates unity if $|r-\tau| \ll r$. The requirement that both nonlinearity and dispersion be weak is satisfied only if $r/d \gg 1$; accordingly, $l/r = (2d^2/r^2)^{\frac{1}{3}} \ll 1$ and $a(r)$ and $l(r)$ are slowly varying *vis-à-vis* $N(z)$ if $z = O(1)$. The description, which is asymptotic, evidently fails for $r \downarrow 0$ and may not remain uniformly valid as $z \downarrow -\infty$ (there is no corresponding difficulty for $z \uparrow \infty$ by virtue of the exponential decay of N in that limit).

The actual amplitude of the wave is aN_1, where $N_1 \equiv N_{\max}$, and the corresponding similarity parameter for the family of solutions described by (1.2) is (the properly scaled measure of nonlinearity/dispersion)

$$aN_1 l^2/d^3 = \tfrac{4}{3}N_1 \equiv \mathscr{S}. \tag{1.4}$$

The energy of the wave is proportional to

$$\rho g(aN_1)^2 lr = \tfrac{1}{2}\mathscr{S}^2 \rho g d^4 \tag{1.5}$$

and therefore is conserved, as anticipated. On the other hand, $N(z)$ proves to be either non-negative ($N \geqslant 0$) or non-positive ($N \geqslant 0$), and the volumetric displacement of the wave, which is proportional to

$$aN_1 lr = \mathscr{S}d^3(r^2/2d^2)^{\frac{1}{3}}, \tag{1.6}$$

is not conserved (although its divergence as $r \uparrow \infty$ with z fixed does not imply divergence as $r \uparrow \infty$ with t fixed).

This last difficulty, together with the limitations cited in the penultimate paragraph, suggests that (1.2) cannot, by itself, represent a complete solution of a well-posed initial-value problem. It may, however, be an asymptotic component of such a solution, in which the mass defect implied by (1.6) is compensated either by some other component or components or by a source (or sink) in some interior domain. It may be compared with the one-dimensional similarity solution of the Korteweg–de Vries equation that was discovered by Berezin & Karpman (1964, 1967) and has since been studied by Ablowitz & Newell (1973), Ablowitz & Segur (1977), Miles (1977*b*) and Rosales (1977). This solution, like the present solution, raises questions about uniform validity in the far region of its dispersive tail; moreover, it may be accompanied by one or more solitons.† The present solution is especially interesting in this context if $\eta \leqslant 0$, since it could appear as an asymptotic component of a solution that also comprises one or more solitary waves of positive elevation [a solitary wave of depression is impossible, since nonlinearity and dispersion then both act to decrease the wave speed, and the wave necessarily breaks up into a decaying wave train, as observed by Scott Russell (1844)]. In any event, nonlinear solutions for two-dimensional wave motion are not easily come by, and I therefore present it as of some intrinsic interest and as a possible stimulus for further research.

[I originally undertook the study of (1.2) in an attempt to describe the tendency of a solitary wave to acquire a dispersive tail in a channel of slowly varying breadth b, for which the joint hypothesis of a balance between weak nonlinearity and dispersion and of conservation of energy implies $a \propto b^{-\frac{2}{3}}$ and $l \propto b^{\frac{1}{3}}$ (Miles 1977*a*). The first peak of the non-negative solution does have a solitary-wave profile (see §5) if

† Indeed, it may be shown that the solution of the KdV equation for any initial displacement of zero volume, which constraint is implied by conservation of mass and the measurement of wave displacement from the surface of static equilibrium, comprises at least one soliton.

$N_1 \gtrsim 3$, but the amplitudes of the subsequent peaks decay only slowly (ultimately like $|z|^{-\frac{1}{4}}$ as $z \downarrow -\infty$) from the maximum, and the solution for $N_1 \gg 1$ is more appropriately described as a slowly varying cnoidal wave.]

The analysis proceeds as follows. In §2, I determine a third-order, nonlinear differential equation for $N(z)$ that admits a one-parameter (N_1 or $\mathscr{S}$) family of solutions that are evanescent at $z = \infty$. I then show that these solutions are either non-negative ($N \geqslant 0$) or non-positive ($N \leqslant 0$). In §3, I show that the transformation $N = \pm F^2(z)$ reduces the third-order equation for N to a second Painlevé equation for F, which is a nonlinear generalization of Airy's equation and also arises in other recent investigations of nonlinear wave motion (see Miles 1977*b* for references). I give representative numerical results in §3 and cite appropriate asymptotic approximations for $|\mathscr{S}| \ll 1$ in §4 and for $|\mathscr{S}| \gg 1$ in §5 (the latter are given explicitly only for $N > 0$). Perhaps the most interesting feature of the solutions for both large and small $\mathscr{S}$ is a cumulative, nonlinear phase shift (relative to the Airy phase of linear theory) in the dispersive tail of the wave.

Finally, in §6, I match the solution for $|\mathscr{S}| \ll 1$ to a solution of the linearized equations, in which nonlinearity is neglected but radial spreading is not assumed to be weak (so that the solution remains valid as $r \downarrow 0$), and introduce a source (or sink) at $r = 0$ to provide for the volumetric defect in the outgoing wave. This suggests a possible experimental configuration, but it is presented primarily as a model that renders the similarity solution physically plausible for sufficiently small $\mathscr{S}$. The matching cannot be carried through for large $\mathscr{S}$ (except for $r \uparrow \infty$, where nonlinear effects are necessarily small by virtue of the exponential decay), however, and the physical significance of the similarity solution is correspondingly more uncertain in that regime.

2. Similarity solution of Boussinesq equations

The Boussinesq equations for a laterally unbounded body of water of quiescent depth d may be placed in the form (Whitham 1967)

$$\xi_t + \tfrac{1}{2}(\nabla\xi)^2 + g\eta = 0 \tag{2.1a}$$

and

$$\eta_t + \nabla\cdot\{(d+\eta)\nabla\xi\} + \tfrac{1}{3}d^3\nabla^4\xi = 0, \tag{2.1b}$$

where ξ and η are the velocity potential at, and the displacement of, the free surface. Eliminating ξ between (2.1*a*, *b*) on the assumptions that the motion is axisymmetric and slowly varying in a reference frame expanding radially with the speed $(gd)^{\frac{1}{2}}$ and that geometrical spreading is weak ($r/d \gg 1$), we obtain the cylindrical Korteweg–de Vries equation (cf. Maxon & Viecelli 1974)

$$\eta_\tau + \eta_r + \tfrac{1}{2}r^{-1}\eta + \tfrac{3}{2}d^{-1}\eta\eta_r + \tfrac{1}{6}d^2\eta_{rrr} = 0 \quad [\tau \equiv (gd)^{\frac{1}{2}}t]. \tag{2.2}$$

Following (1.2), we posit the similarity solution

$$\eta(r,t) = \tfrac{4}{3}d(2r/d)^{-\frac{2}{3}}N(z), \quad z = (2d^2r)^{-\frac{1}{3}}(r-\tau) \tag{2.3a,b}$$

in (2.2) to obtain

$$N''' + 12NN' - 4zN' - 2N = 0 \tag{2.4}$$

to within $1 + O(d^{\frac{2}{3}}/r^{\frac{2}{3}})$. We seek the solution of (2.4) subject to the null conditions

$$zN, N', N'' \to 0 \quad (z \to \infty). \tag{2.5}$$

We remark that the assumption of inward, rather than outward, wave propagation requires only that the sign of τ be changed in (2.2) and (2.3*b*) and leaves (2.4) unchanged.

Integrating (2.4) once, multiplying the result by $2N'$, integrating again, and invoking (2.5), we obtain

$$N'' = 4zN - 6N^2 + 2\int_z^\infty N dz \tag{2.6}$$

and

$$N'^2 = 4N\left(zN - N^2 + \int_z^\infty N dz\right). \tag{2.7}$$

Now suppose that $N > 0$ and $o(1/z)$ as $z \uparrow \infty$. It then may be inferred from (2.6) and (2.7) that N has a turning point, a first maximum and a first zero at $z = z_2$, z_1 and z_0, respectively, where $z_2 > z_1 > z_0$, and that the first minimum of N coincides with the first zero ($N' = 0$ and $N'' > 0$ for $N = 0$ at $z = z_0$) and similarly for all following zeros; accordingly $N \geqslant 0$ for all z. It also follows that minima are possible only if $N = 0$, since $N' = 0$ and $N > 0$ imply $N'' < 0$ and therefore correspond to a maximum.

The situation is more delicate if $N < 0$ as $z \uparrow \infty$, since the existence of a turning point then can be definitely established only for sufficiently weak nonlinearity (see §3); however, if N does have zeros, they must coincide with its maxima, and $N \leqslant 0$ for all z.

3. Reduction to Painlevé equation

A consideration of the known solutions of the linear differential equation obtained by neglecting $12NN'$ in (2.4), together with the hypothesis that N is either non-negative or non-positive, suggests the transformation

$$N = \pm F^2(z), \tag*{(3.1)$_\pm$}$$

where, here and subsequently, alternative signs and subscripts are vertically ordered. The transformed equation is

$$F(F'' - zF)' + 3F'(F'' - zF) \pm 12F^3F' = 0. \tag*{(3.2)$_\pm$}$$

Multiplying (3.2) by F^2, integrating, invoking the null conditions (2.5) at $z = \infty$, and dividing the result by F^3, we obtain the Painlevé equation

$$F'' - zF \pm 2F^3 = 0. \tag*{(3.3)$_\pm$}$$

That solution which vanishes for $z \uparrow \infty$ has the asymptotic approximations

$$F(z) \sim A\,\mathrm{Ai}(z) \quad (z \gg 2F^2) \tag{3.4a}$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}}Az^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right)\{1 + O(z^{-\frac{3}{2}})\} \quad (z \uparrow \infty). \tag{3.4b}$$

We may assume $A > 0$, since (3.1) and (3.3) are invariant under the transformation $F \to -F$.

It follows from (2.3), (3.1) and (3.4*b*) that

$$\eta \sim \pm(A^2d^2/3\pi)\{2r(r-\tau)\}^{-\frac{1}{2}}\exp\{-\tfrac{4}{3}(2d^2r)^{-\frac{1}{2}}(r-\tau)^{\frac{3}{2}}\} \quad (r \gg r - \tau \gg l). \tag*{(3.5)$_\pm$}$$

We remark that (3.5) is essentially a linear approximation (nonlinear effects are insignificant for $z \gg 1$); see §6.

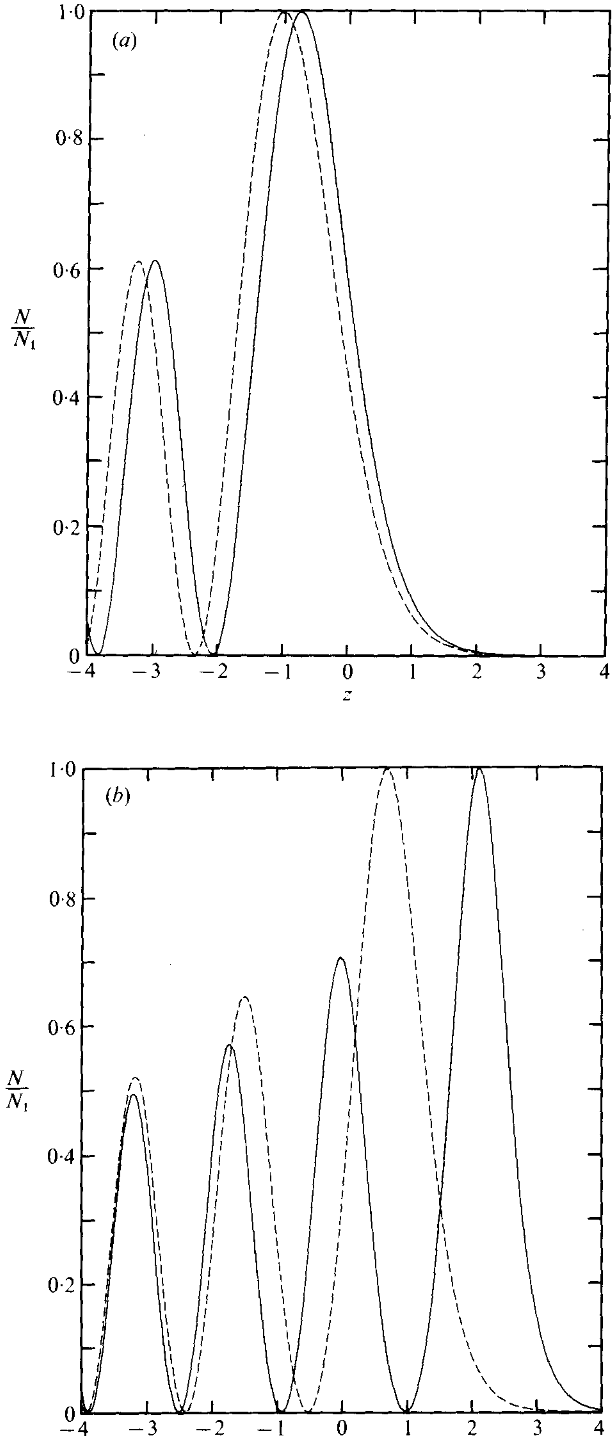

FIGURE 1. The normalized profiles $N/N_1 = (F_+/F_1)^2$ for (a) $A = 0$ (- - - -), 1 (——) and (b) $A = 10$ (- - -), 10^2 (——), as determined by the numerical solution of $(3.3)_+$ and (3.4).

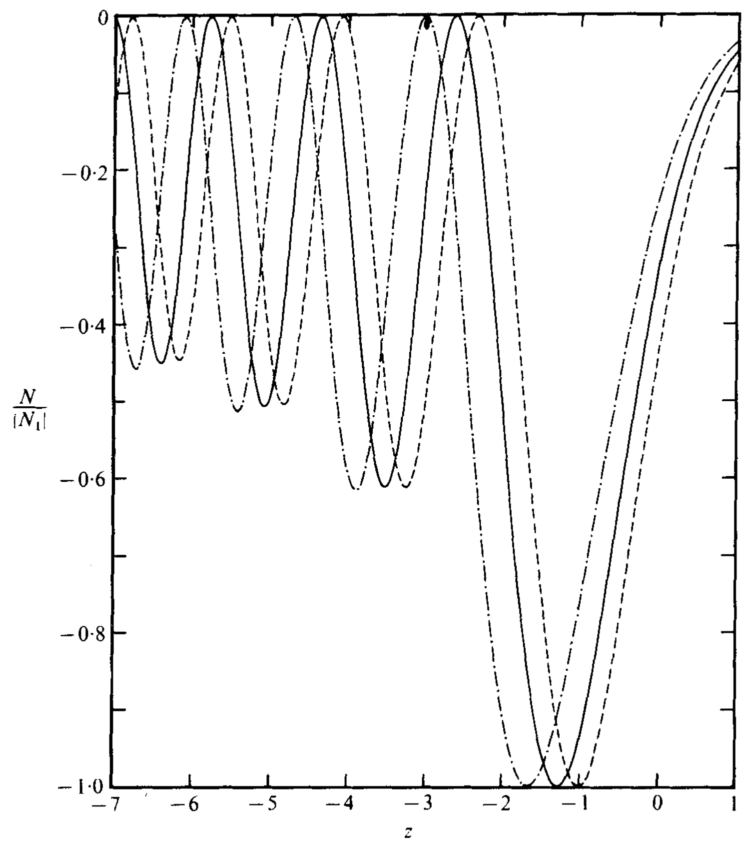

FIGURE 2. The normalized profiles $N/|N_1| = -(F_-/F_1)^2$ for $A = 0$ (---), 0·7 (——) and 0·9 (—·—), as determined by the numerical solution of $(3.3)_-$ and (3.4).

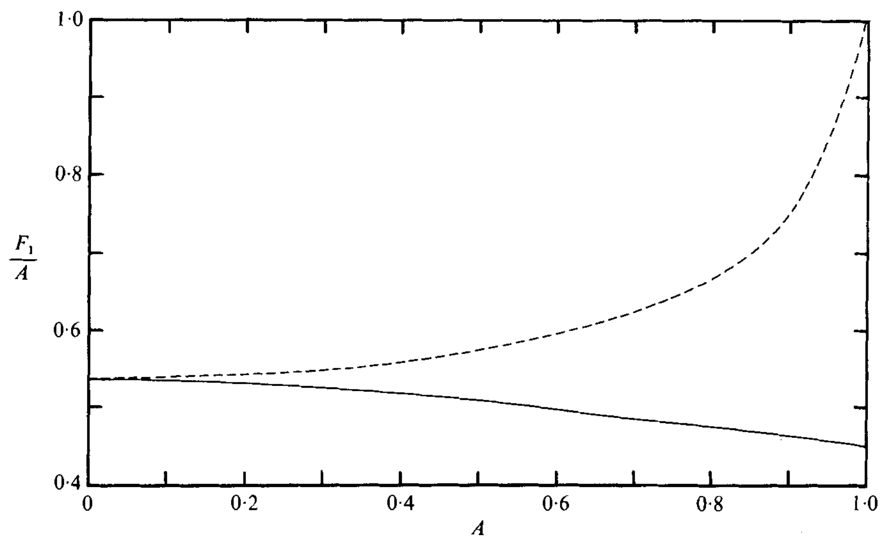

FIGURE 3. The parameter F_1/A for F_+ (——) and F_- (---); see last paragraph in §3.

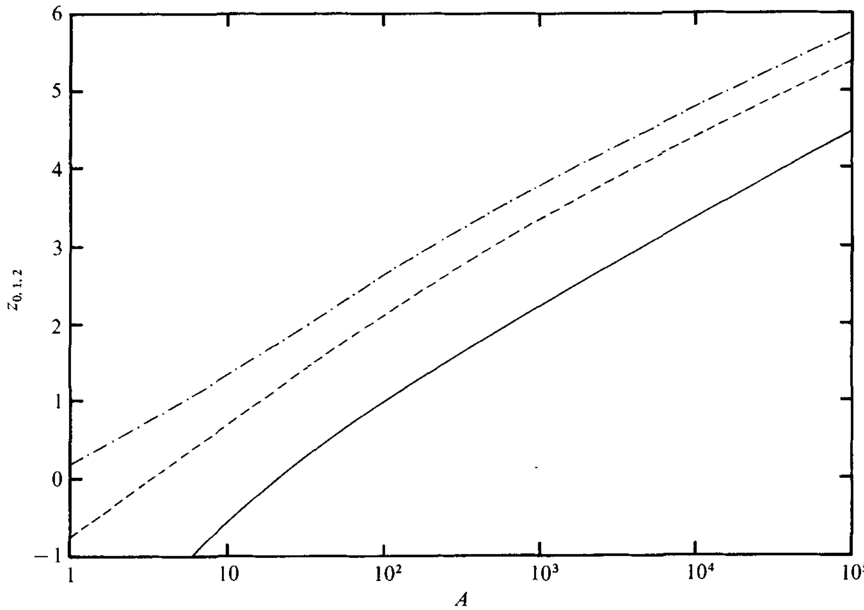

FIGURE 4. The largest zeros of F_+, F'_+ and F''_+: z_0 (——), z_1 (– – –) and z_2 (—·—).

The numerical integration of $(3.3)_+$ in the direction of decreasing z, using either (3.4*a*) or (3.4*b*) to obtain starting values of F and F' at a sufficiently large value of z, is both stable and economical and yields a one-parameter family of solutions with A as the family parameter. Representative results are plotted in figures 1, 3 and 4. See Miles (1977*b*) for further results.

The corresponding integration of $(3.3)_-$ and (3.4) is possible only for $0 < A < 1$, since the solution is singular at some finite real value of z if $A > 1$. Representative results are plotted in figures 2 and 3. See Miles (1977*b*) and Rosales (1977) for further results.

The parameter F_1/A, in terms of which [see (1.4)]

$$\mathscr{S} = \pm\tfrac{4}{3}(F_1/A)^2A^2, \tag{3.6$_\pm$}$$

is plotted in figure 3. [*Note added in press*: I have since found that $F_1/\mathscr{A}_\pm$, where $\pi\mathscr{A}_\pm^2 = \pm\ln(1\pm A_\pm^2)$, varies by less than 10% over $0 < A_\pm < 10^6/0{\cdot}999$.]

4. The limit $A \downarrow 0$

The solution of $(3.3)_\pm$ and (3.4) may be developed as a perturbation about the linear approximation $A\,\mathrm{Ai}(z)$ if $A \lesssim 1$. The interesting regime is $-z \gg 1$, wherein the straightforward expansion in powers of A is not uniformly valid but can be so rendered to obtain (Miles 1977*b*; the approximation is within a few per cent of the results of numerical integration for $z < -1$ and either $A \lesssim 1$ for F_+ or $A \lesssim 0{\cdot}7$ for F_-)

$$F(z) = \pi^{-\frac{1}{2}}A(1 \mp \tfrac{1}{4}A^2)(-z)^{-\frac{1}{4}}\sin(\chi \pm A^2\Lambda) + O(A^4, Az^{-\frac{7}{4}}) \quad (z \downarrow -\infty), \tag{4.1$_\pm$}$$

where

$$\chi = \tfrac{2}{3}(-z)^{\frac{3}{2}} + \tfrac{1}{4}\pi = \tfrac{2}{3}(2d^2r)^{-\frac{1}{2}}(\tau - r)^{\frac{3}{2}} + \tfrac{1}{4}\pi \tag{4.2}$$

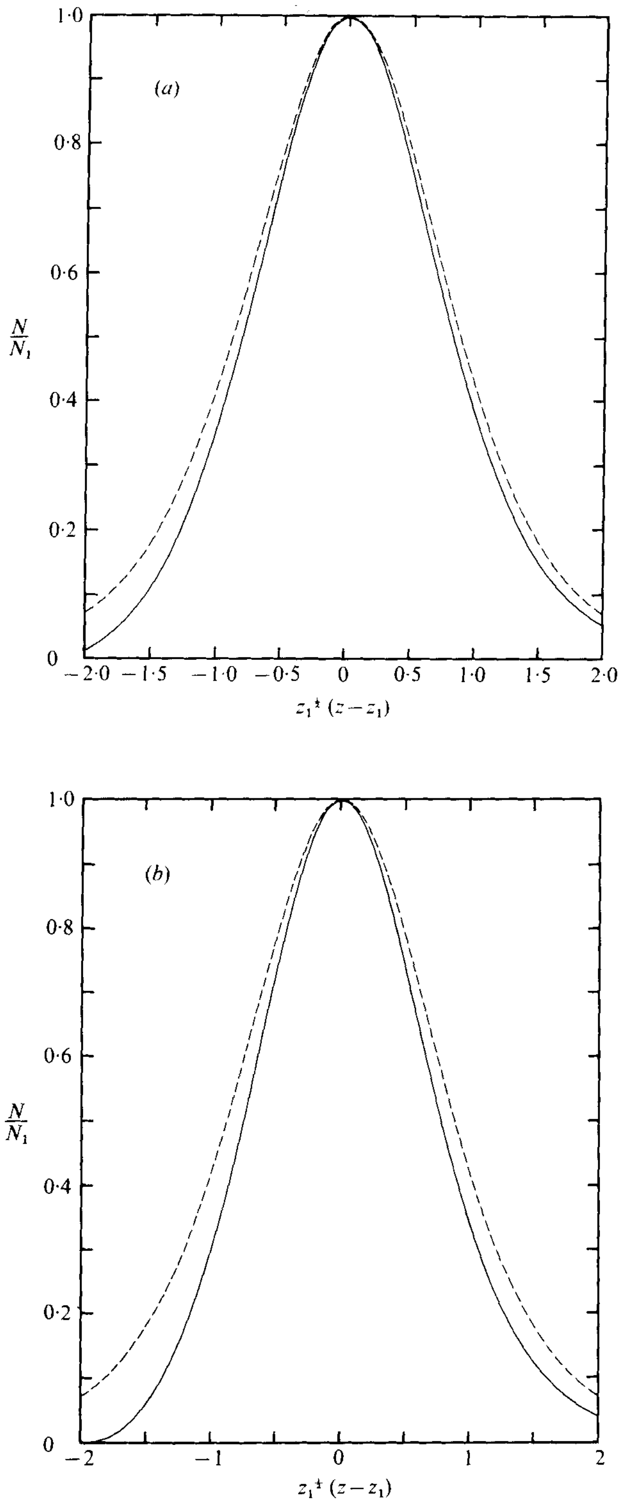

Figure 5. Comparison of the normalized profile $N/N_1 = (F_+/F_1)^2$ (——) with the solitary-wave profile implied by (5.1) (– – –) for (*a*) $z_1 = 5{\cdot}4$ ($A = 10^5$) and (*b*) $z_1 = 3{\cdot}3$ ($A = 10^3$).

is the usual Airy phase, and

$$\Lambda = (4/3\pi)+(3/4\pi)\ln(-z) = (4/3\pi)+(3/4\pi)\ln\{(2d^2r)^{-\frac{1}{3}}(\tau-r)\} \tag{4.3}$$

is the cumulative (normalized) phase shift induced by nonlinearity. Substituting $N = \pm F^2$ from (4.1) into (2.3), we obtain [cf. (3.5)]

$$\eta \sim \pm(4A^2d^2/3\pi)(1\mp\tfrac{1}{2}A^2)\{2r(\tau-r)\}^{-\frac{1}{2}}\sin^2(\chi\pm A^2\Lambda) \quad (r \gg \tau-r \gg l). \quad (4.4)_\pm$$

5. The limit $A \uparrow \infty$

The asymptotic solution of $(3.3)_+$ and (3.4) for $A\uparrow\infty$ in the neighbourhood of the wave-front maximum, $F_1 \equiv F(z_1)$, is described by (Miles 1977*b*)

$$F_+(z) \sim z_1^{\frac{1}{2}}\operatorname{sech}\{z_1^{\frac{1}{4}}(z-z_1)\}+O(z_1^{-1}) \quad [z_1\uparrow\infty, \quad z-z_1 = O(z_1^{\frac{1}{4}})], \tag{5.1}$$

the matching of which to (3.4*b*) yields

$$A \sim 4\pi^{\frac{1}{2}}z_1^{\frac{1}{4}}\exp(\tfrac{2}{3}z_1^{\frac{3}{2}}) \quad (z_1\uparrow\infty). \tag{5.2}$$

The normalized profile F^2/F_1^2 given by (5.1), which corresponds to that of a one-dimensional solitary wave, is compared with the results of the numerical integration in figure 5.

The oscillatory tail of F_+ may be described rather accurately by a slowly varying cnoidal wave if $A \gtrsim 10$ (Miles 1977*b*), and has the limiting form

$$F_+ \sim \mathscr{A}(-z)^{-\frac{1}{4}}\sin[\tfrac{2}{3}(-z)^{\frac{3}{2}}+\mathscr{A}^2\{\tfrac{3}{4}\ln(-z)-\ln\mathscr{A}+1{\cdot}683\}] \quad (-z \gg z_1 \gg 1), \tag{5.3}$$

where
$$\mathscr{A} = \pi^{-\frac{1}{2}}\{\ln(1+A^2)\}^{\frac{1}{2}} \sim \pi^{-\frac{1}{2}}\{\tfrac{4}{3}z_1^{\frac{3}{2}}+\ln(16\pi z_1^{\frac{1}{2}})\} \quad (z_1\uparrow\infty). \tag{5.4}$$

Similar results are available for F_- (Miles 1977*b*).

6. The linear regime

The volumetric defect implied by the similarity solution (2.3) suggests the existence of a source in some interior domain in which (2.2) is invalid. The lateral dimensions of this domain are irrelevant (in the present context) in so far as they are sufficiently small, and we therefore assume that the source is concentrated at $r = 0$ (a more detailed analysis for a distributed source reveals that only its total volumetric input is significant for the asymptotic solution that ultimately evolves in $r \gg l$). We also assume that, at least for $A \ll 1$ (see below), the nonlinear terms can be neglected during the initial evolution, so that we may replace (2.1) by

$$\xi_t+g\eta = 0 \tag{6.1a}$$

and

$$\eta_t+d\nabla^2\xi+\tfrac{1}{3}d^3\nabla^4\xi = Q\delta(\mathbf{r}), \tag{6.1b}$$

where $\delta(\mathbf{r})$ is the two-dimensional delta function, and Q is the volumetric rate of the source. We choose the initial conditions

$$\xi = \eta = 0 \quad (t = 0) \tag{6.2}$$

and proceed on the hypothesis that Q is independent of t. The required solution, which may be obtained through a Hankel transformation with respect to r and a Laplace transformation with respect to t, is given by

$$\eta \sim \mathscr{Q} \int_0^\infty J_0(kr) \sin\{k(1-\tfrac{1}{6}k^2d^2)\tau\} dk, \tag{6.3}$$

where

$$\mathscr{Q} = Q/(2\pi c_0), \quad c_0 = (gd)^{\frac{1}{2}}, \tag{6.4a,b}$$

and $O(k^2d^2)$ terms have been neglected in the amplitude but retained in the phase (dispersion is finite but weak).

The solution (6.3) can be matched to the similarity solution (2.3) in $r \gg r-\tau \gg l$. Letting $r \uparrow \infty$, invoking

$$J_0(kr) \sim (\tfrac{1}{2}\pi kr)^{-\frac{1}{2}} \cos(kr - \tfrac{1}{4}\pi) \quad (kr \uparrow \infty) \tag{6.5}$$

to obtain

$$\eta \sim \mathscr{Q}(2\pi r)^{-\frac{1}{2}} \int_0^\infty [\cos\{k(r-\tau) + \tfrac{1}{6}k^3d^2\tau + \tfrac{1}{4}\pi\} - \cos\{k(r+\tau) - \tfrac{1}{6}k^3d^2\tau + \tfrac{1}{4}\pi\}] k^{-\frac{1}{2}} dk, \tag{6.6}$$

carrying out a saddle-point approximation, and approximating τ by r except in $r-\tau$, we obtain

$$\eta \sim \tfrac{1}{2}\mathscr{Q}\{2r(r-\tau)\}^{-\frac{1}{2}} \exp\{-\tfrac{4}{3}(2d^2r)^{-\frac{1}{2}}(r-\tau)^{\frac{3}{2}}\} \quad (r \gg r-\tau \gg l). \tag{6.7}$$

Comparing (6.7) to (3.5), we obtain

$$A^2 = \tfrac{3}{2}\pi|\mathscr{Q}|/d^2 = \tfrac{3}{4}|Q|/c_0d^2, \tag{6.8}$$

where $Q \gtrless 0$ corresponds to $\eta \gtrless 0$.

The corresponding approximation to (6.6) for $r \gg \tau - r \gg l$ is dominated by the contributions from the singular point at $k = 0$ and the point of stationary phase at $kd = \{2(\tau-r)/\tau\}^{\frac{1}{2}}$. Following the procedure described by Erdélyi (1956, §2.9), approximating τ by r except in $\tau - r$, and introducing χ from (4.2b), we obtain

$$\eta \sim \mathscr{Q}\{\tfrac{1}{2}r(\tau-r)\}^{-\frac{1}{2}} \sin^2\chi \quad (l \ll \tau - r \ll r), \tag{6.9}$$

which matches (4.4) in the limit $A \downarrow 0$ if (6.8) holds. On the other hand (and not unexpectedly), it cannot be matched to the profile implied by (5.3).

If $\tau \gg r$, so that the inequality $\tau - r \ll r$ is violated, the integral in (6.3) has no point of stationary phase, and the asymptotic approximation is dominated by the end point at $k = 0$. The resulting approximation is

$$\eta \sim \mathscr{Q}(\tau^2 - r^2)^{-\frac{1}{2}} \quad (\tau \gg r, d), \tag{6.10}$$

which provides a description of the ultimate decay of the wave in the neighbourhood of $r = 0$ (the retention of r^2 *vis-à-vis* τ^2 in the radical is consistent with the neglect of higher-order terms in $1/\tau$ if $r \gg d$). It is curious that $\eta \sim \mathscr{Q}/\tau$ at $r = 0$ despite the continuing action of the source at that point.

This work was partially supported by the Physical Oceanography Division, National Science Foundation, NSF Grant OCE74-23791, and by the Office of Naval Research under Contract N00014-76-C-0025. I am indebted to Professor Whitham for pointing out that the similarity solution (1.2) may be negative as well as positive and for emphasizing the difficulties associated with the volumetric defect implied by either the non-negative or non-positive character of the wave.

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